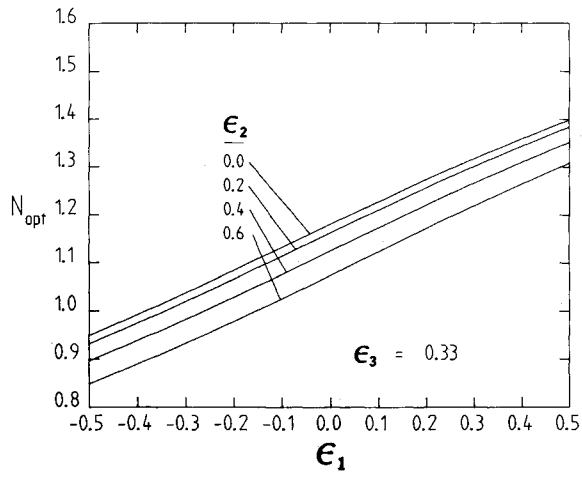

Fig. 4 Optimum N for turbulent natural convection.

by Kraus,[12,13] in terms of modified Bessel function and is essentially identical to the numerical solution given here.

Figure 2c compares the numerical and perturbation results for $\epsilon_3 = 0.25$ and 0.33 while holding ϵ_1 and ϵ_2 to zero. The maximum difference shown for this case is 2.0%.

Finally, Fig. 2d compares the numerical and perturbation solutions for $\epsilon_1 = 0.5$, $\epsilon_2 = 0.6$, and $\epsilon_3 = 0.33$. These values represent a worst case in which the perturbation parameters are maximum in the practical application range. The maximum error in this case is 7.3%.

The fin efficiency, η, is readily calculated by dividing the heat transfer q through the base by the ideal heat transfer rate. The result is

$$\eta = 1/N^2\{(1+\epsilon_1)N\tanh N + (\epsilon_1/3)N\tanh N(\tanh^2 N - 3) + (\epsilon_2/8)\operatorname{sech}^2 N(2N^2 + 1 - \cosh 2N - \sinh 2N) + \epsilon_3[C_1 N\sinh N + C_2(\sinh N + N\cosh N) - C_3(84N^2 + 41/3\,N^4 + 37/60\,N^6 + 1/70\,N^8)]\} \quad (16)$$

The fin can be optimized with respect to volume by finding the values of b and δ that yield the maximum heat transfer for a given set of the perturbation parameters. Upon performing the derivatives $dq/db + dq/d\delta = 0$ and solving for N, the optimum dimensions can be calculated. Given in Figs. 3 and 4 are the optimum values of N as a function of ϵ_1 and ϵ_2 for laminar and turbulent conditions, respectively. These curves may be compared with the result given by Krane,[14] in which fin taper and a temperature-dependent heat transfer coefficient were not considered. When $\epsilon_2 = \epsilon_3 = 0$, Krane's curve is duplicated. Nonzero values of ϵ_2 and ϵ_3 shift N_{opt} downward from the curve in Ref. 14.

Conclusion

A first-order, three-parameter perturbation expansion has been shown to predict accurately the temperature distribution in longitudinal tapered fins with temperature-dependent thermal conductivity and heat transfer coefficient. The solution was carried out to second order, but the accuracy was not appreciably improved over the result given by the first-order expansion. Hence, the first-order solution is sufficient for fin design. Fin efficiency and optimum dimensions have been given as a function of the perturbation parameters. These results should prove to be useful in the design of tapered fins.

References

[1]Aziz, A. and Na, T.Y., *Perturbation Methods in Heat Transfer,* Hemisphere, Washington, DC, 1984, pp. 14–15, 44–46.

[2]Aziz, A. and Enamul Huq, S.M., "Perturbation Solution for Convecting Fin with Variable Thermal Conductivity," *ASME Journal of Heat Transfer,* May 1975, pp. 300–301.

[3]Aziz, A. and Benzies, J.Y., "Application of Perturbation Techniques to Heat-Transfer Problems with Variable Thermal Properties," *International Journal of Heat and Mass Transfer,* Vol. 19, 1976, pp. 271–276.

[4]Aziz, A., "Perturbation Solution for Convective Fin with Internal Heat Generation and Temperature Dependent Thermal Conductivity," *International Journal of Heat and Mass Transfer,* Vol. 20, 1977, pp. 1253–1255.

[5]Aziz, A. and Na, T.Y., "Periodic Heat Transfer in Fins with Variable Thermal Parameters," *International Journal of Heat and Mass Transfer,* Vol. 24, 1981, pp. 1397–1404.

[6]Bilenas, J.A. and Jij, L.M., "A Perturbation Solution for Fins with Conduction, Convection, and Radiation Interaction," *AIAA Journal,* Vol. 8, 1970, pp. 168–169.

[7]Holman, J.P., *Heat Transfer,* 4th ed., McGraw-Hill, New York, 1976, p. 253.

[8]Stachiewicz, J.W., "Effect of Variation of Local Film Coefficient on Fin Performance," *ASME Journal of Heat Transfer,* Vol. 91, 1969, pp. 21–26.

[9]Bellman, R., *Perturbation Techniques in Mathematics, Physics, and Engineering,* Holt, Rinehart and Winston, New York, 1964, pp. 15–17.

[10]Arunachalam, M. and Seeniraj, V., Comments on "Application of Perturbation Techniques to Heat Transfer Problems with Variable Thermal Properties," *International Journal of Heat and Mass Transfer,* Vol. 20, 1977, pp. 803–804.

[11]Snider, A.D. and Kraus, A.D., "Recent Developments in the Analysis and Design of Extended Surface," *ASME Journal of Heat Transfer,* Vol. 105, May 1983, pp. 302–306.

[12]Kraus, A.D., *Extended Surfaces,* Spartan, Baltimore, MD, 1964, pp. 52–55, 76–78.

[13]Kraus, A.D., *Analysis and Evaluation of Extended Surface Thermal Systems,* Hemisphere, Washington, D.C., 1982, pp. 100–104.

[14]Krane, R.J., "Discussion of Perturbation Solution for Convecting Fin with Variable Thermal Conductivity," *ASME Journal of Heat Transfer,* Nov. 1976, pp. 685–686.

Correlation of the Gap Conductance Integral for Conforming Rough Surfaces

K. J. Negus* and M. M. Yovanovich†
University of Waterloo, Waterloo, Ontario, Canada

Introduction

STEADY-STATE heat transfer across the interface of two contracting solid bodies is usually accompanied by a measurable temperature drop due to the thermal resistance to heat flow in the interface region. If the temperature drop ΔT_c at the interface is obtained by extrapolation from regions "far" from the interface and Q/A_a is the steady-state heat flux based on the apparent contact area, then the joint conductance or contact coefficient of heat transfer is defined as

$$h_j = (Q/A_a)/\Delta T_c \quad (1)$$

Considerable research has been undertaken in recent years to develop models to predict joint conductance and to experimentally verify these models as discussed in detail by

Received Dec. 11, 1986; revision received May 1, 1987.

*Graduate Research Assistant, Microelectronics Heat Transfer Laboratory, Department of Mechanical Engineering. Student Member AIAA.

†Professor, Microelectronics Heat Transfer Laboratory, Department of Mechanical Engineering. Associate Fellow AIAA.

Yovanovich.[1,2] Thermal contact resistance is an important consideration for many aerospace systems and is becoming increasingly important in the cooling of microelectronic equipment.[3]

Conforming Rough Surfaces Model

The theory of conforming rough surfaces as proposed by Cooper et al.[4] is based on a Gaussian distribution of the surface roughness parameters. For interfaces where an interstitial gas is present, it is convenient to deal with the contact and gap contributions to heat transfer separately. Since the contact and gap heat transfer rates are approximately independent for most practical problems,[2] then for negligible radiative heat transfer the joint conductance is modeled as

$$h_j = h_c + h_g \tag{2}$$

where h_c is the contact conductance for the interface under vacuum conditions and h_g is the gap conductance for heat transfer across the gap only.

The theory behind the contact conductance model has been developed in detail in several references. For most practical problems, h_c can be estimated accurately by the correlation

$$h_c = \frac{1.25 m k_s}{\sigma}\left(\frac{P}{H}\right)^{0.95} \tag{3}$$

where P is the apparent pressure on the interface and H the hardness of the softer material. If the subscripts 1 and 2 are understood to refer to surfaces 1 and 2 in contact, then the harmonic mean thermal conductivity, effective rms surface roughness, and effective absolute surface slope are defined, respectively, as

$$k_s = 2k_1 k_2/(k_1 + k_2) \tag{4}$$

$$\sigma = \sqrt{\sigma_1^2 + \sigma_2^2} \tag{5}$$

$$m = \sqrt{m_1^2 + m_2^2} \tag{6}$$

Equation (3) has been shown to agree within $\pm 1.5\%$ of the exact theoretical results for the range $10^{-6} \le P/H \le 10^{-2}$ for both similar and dissimilar materials.

Heat transfer across the gap formed by two conforming rough surfaces in contact is difficult to analyze because of the complexity of the local geometry which determines whether the local heat transfer should be modeled as continuum, slip, or rarefied. Yovanovich et al.[5] have overcome these difficulties by again using a Gaussian distribution to describe the surface roughness characteristics and a local gas conductivity model as suggested by Kaganer.[6] Their derivation gives

$$h_g = \frac{k_g}{\sigma} I_g \tag{7}$$

where k_g is the thermal conductivity of the gas under continuum conditions and I_g is the gap conductance integral defined by

$$I_g = \frac{1}{\sqrt{2\pi}} \int_0^\infty \frac{\exp[-(Y/\sigma - u)^2/2}{u + M/\sigma}\, du \tag{8}$$

In this expression, Y/σ is the relative mean plane separation between the two contacting solids and is given accurately by the correlation

$$Y/\sigma = 1.184[-\ln(3.132P/H)]^{0.547} \tag{9}$$

The parameter M is given by

$$M = \alpha\beta\Lambda \tag{10}$$

The accommodation parameter α in Eq. (10) is defined as

$$\alpha = \frac{2-\alpha_1}{\alpha_1} + \frac{2-\alpha_2}{\alpha_2} \tag{11}$$

where α_1 and α_2 are the accommodation coefficients at the solid gas interfaces. The fluid property β is defined by

$$\beta = \frac{2\gamma}{Pr(\gamma + 1)} \tag{12}$$

where γ is the ratio of specific heats and Pr the gas Prandtl number. The mean free path Λ of the gas molecules is given in terms of $\Lambda_{g,\infty}$ and the mean free path at standard temperature and pressure (STP) by the expression

$$\Lambda = \Lambda_{g,\infty} \frac{T_g}{T_{g,\infty}} \frac{P_{g,\infty}}{P_g} \tag{13}$$

where T_g and P_g are the gas temperature and pressure in the gap and $T_{g,\infty}$ and $P_{g,\infty}$ represent STP.

At this point the joint conductance h_j can be predicted for a given situation by assembling the appropriate thermophysical data and applying the given equations. Furthermore, the extensive experimental investigations of Hegazy[7] have shown the conforming rough surfaces model to give good predictions of the joint conductance over a wide range of practical problems. Unfortunately though, no closed form expression has yet been found for I_g, and thus Eq. (8) must be integrated numerically. Numerical integration of the gap conductance integral can be performed rapidly even on a microcomputer but makes hand calculations and quick order-of-magnitude estimates for h_j impractical.

Correlations for the Gap Conductance Integral

Yovanovich[1] has suggested that the gap conductance integral can be approximated by the simple expression

$$I_g \approx \frac{1}{Y/\sigma + M/\sigma} \tag{14}$$

Although this expression is accurate within 10% for large values of Y/σ and M/σ, Eq. (14) significantly underpredicts the gap conductance by 50–100% for small values of Y/σ and M/σ. A new correlation to overcome this problem is now proposed by modifying Eq. (14) with a correction factor f_g such that

$$I_g = \frac{f_g}{Y/\sigma + \mathrm{M}/\sigma} \tag{15}$$

Exact values of f_g were generated by numerically integrating Eq. (8) over a wide range of Y/σ and M/σ. By examining the trends and limit points in this data, simple approximate expressions for f_g have been derived as

$$f_g = 1.063 + 0.0471\left(4 - \frac{Y}{\sigma}\right)^{1.68}\left(\ln\frac{\sigma}{M}\right)^{0.84} \tag{16}$$

for $2 \le Y/\sigma \le 4$ and $0.01 \le M/\sigma \le 1$, and

$$f_g = 1 + 0.06\left(\frac{\sigma}{M}\right)^{0.8} \tag{17}$$

Table 1 Predicted and measured conductances for example problem

$10^3 \frac{P}{H}$	Conductances ($W/m2K$) Predicted h_c	Predicted h_g	Predicted h_j	Measured h_j
0.165	170	1800	1970	2300
0.265	260	1880	2140	2430
0.364	360	1940	2300	2560
0.498	480	2010	2490	2800
0.651	620	2080	2700	3030
0.809	760	2140	2900	3240
1.130	1050	2230	3280	3690
1.459	1330	2310	3640	4070
1.788	1620	2380	4000	4570
2.091	1880	2440	4320	4910
2.577	2290	2520	4810	5500
3.162	2780	2600	5380	6220

for $2 \le Y/\sigma \le 4$ and $1 \le M/\sigma < \infty$. The maximum error associated with the simple correlations of Eqs. (16) and (17) is about 2% with this respect to the numerically integrated results.

Example Problem

The agreement between the simple correlations summarized here and real experimental data can be illustrated by considering an example test case from the recent work of Hegazy.[7] In particular, the test results for a pair of 304 Stainless Steel specimens contacting in a nitrogen environment at 570 Torr are summarized in Table 1. The predicted conductances obtained from the simple correlations of this work are also summarized in Table 1 based upon the thermophysical data supplied by Hegazy.[7] Note that $M/\sigma \approx 0.093$ in this case for all the results reported in Table 1. It is evident, at least for this example, that the agreement between the predicted and experimental conductances is excellent.

Conclusions

New correlations for the gap conductance integral in the conforming rough surface model have been presented. Accurate estimations of the thermal joint conductance of many new practical contact problems can now be made quickly by combining these new correlations with the simple contact conductance correlations also provided in this Note.

Acknowledgments

The authors acknowledge the financial support of this work provided by the Natural Sciences and Engineering Research Council of Canada under operating grant A7455 for Dr. Yovanovich and from a Postgraduate Scholarship for Mr. Negus.

References

[1]Yovanovich, M. M., "Thermal Contact Correlations," *Progress in Astronautics and Aeronautics: Spacecraft Radiative Transfer and Temperature Control*, Vol. 83, edited by T. E. Horton, AIAA, New York, 1982, pp. 83-95.

[2]Yovanovich, M. M., "Recent Developments in Thermal Contact, Gap and Joint Conductance Theories and Experiment," keynote address at 8th International Heat Transfer Conference, San Francisco, CA, Aug. 17-22, 1986.

[3]Antonetti, V. W. and Yovanovich, M. M., "Thermal Contact Resistance in Microeletronic Equipment," *Thermal Management Concepts in Microelectronics Packaging*, ISHM Technical Monograph, 1984, pp. 193-214.

[4]Cooper, M., Mikic, B. B., and Yovanovich, M. M., "Thermal Contact Conductance," *International Journal of Heat and Mass Transfer*, Vol. 12, Jan. 1969, pp. 279-300.

[5]Yovanovich, M. M., DeVaal, J., and Hegazy, A. H., "A Statistical Model to Predict Thermal Gap Conductance Between Conforming Rough Surfaces," AIAA Paper 82-0888, June 1982.

[6]Kaganer, M. G., *Thermal Insulation in Cryogenic Engineering*, translated from Russian by M. Moscona, Israel Program for Scientific Translations, Jerusalem, 1969.

[7]Hegazy, A. H., "Thermal Joint Conductance of Confirming Rough Surfaces: Effect of Surface Micro-Hardness Variations," Ph.D. Thesis, Dept. of Mechanical Engineering, University of Waterloo, Canada, 1985.

Steady Conjugate Heat Transfer in Fully Developed Laminar Pipe Flows

J.C. Kuo* and T.F. Lin†
National Chiao Tung University, Hsinchu, Taiwan, Republic of China

Introduction

FORCED convective heat transfer in a laminar pipe flow has long been recognized as a basic heat-transfer problem and has been extensively studied in the past. Recently, the effects of the conduction heat transfer in the pipe wall on the convection heat transfer in the flow have received considerable attention.[1-6] Close examination of these recent investigations reveals that either the radial thermal resistance of the wall is neglected or the flow is assumed to be uniform without any rigorous justification.

In the present study, we consider the conjugate heat transfer in laminar flow through a long circular pipe that is directly heated over a finite length. Both axial and radial heat conduction in the pipe wall and fluid are accounted for in the analysis for the entire pipe (directly heated and unheated). Emphasis is placed on the comparison between the results of the present study and those neglecting the radial wall resistance.[2] An exponential finite-difference scheme is employed to solve the coupled energy equations.

Analysis

The system to be studied corresponds to a fluid flowing in an infinitely long circular pipe ($-\infty < x < \infty$). A uniform heat flux q''_{wo} is applied at the outer surface of the pipe over a finite length ($0 \le x \le \ell$). The upstream ($-\infty < x < 0$) and downstream ($\ell < x < \infty$) regions of the heating zone are thermally well insulated, and the flow enters the pipe with uniform velocity u_e and uniform temperature T_e from the far upstream region ($x \to -\infty$). In this study we are interested in the thermal interactions between the conduction heat transfer in the pipe wall and the convection heat transfer in the fluid through the fluid wall interface.

To simplify the analysis, the thermophysical properties of the fluid are assumed to be temperature-independent, and the flow is hydrodynamically fully developed in the region where significant heat transfer is present. Accordingly, the steady conjugate heat transfer in the system considered can be described by the following basic equations in dimensionless form:

Energy equation for the fluid:

$$Pe(1-\eta^2)\frac{\partial \theta_f}{\partial \xi} = \frac{1}{\eta}\frac{\partial}{\partial \eta}\left(\eta \frac{\partial \theta_f}{\partial \eta}\right) + \frac{\partial^2 \theta_f}{\partial \xi^2} \qquad (1)$$

Received Jan. 15, 1987; revision received July 30, 1987.

*Graduate Student. Department of Mechanical Engineering.
†Associate Professor. Department of Mechanical Engineering.